Marginal and density atomic Wehrl entropies for the Jaynes–Cummings model

2008 J. Phys. A: Math. Theor. 41 035303

(http://iopscience.iop.org/1751-8121/41/3/035303)

J. Phys. A: Math. Theor. **41** (2008) 035303 (10pp) doi:10.1088/1751-8113/41/3/035303

Marginal and density atomic Wehrl entropies for the Jaynes–Cummings model

Faisal A A El-Orany

Department of Mathematics and Computer Science, Faculty of Science, Suez Canal University, 41522 Ismailia, Egypt

E-mail: el-orany@yahoo.com

Received 24 August 2007, in final form 10 November 2007
Published 4 January 2008
Online at stacks.iop.org/JPhysA/41/035303

Abstract
In this paper, we develop the notion of the marginal and density atomic Wehrl entropies for a two-level atom interacting with the single mode field, i.e. the Jaynes–Cummings model. For this system we show that there are relationships between these quantities and both the information entropies and the von Neumann entropy.

PACS numbers: 42.50.Dv, 42.50.−p

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The entanglement represents one of the most remarkable features of quantum mechanics. For an entangled system it is impossible to factorize its state in a product of independent states to describe its parts. In recent years, the entanglement has been recognized as a resource for quantum-information processing [1–3]. Various types of experiments have been performed on the entanglement in the quantum systems, e.g. long-distance entanglement [4], ion-photon entanglement [5], many photons entanglement [6], etc. For a recent review, the reader can consult [7].

Generally, the entanglement in quantum systems is investigated by means of the entropy [8]. There are various definitions for the entropy including the von Neumann entropy [8], the relative entropy [9], the generalized entropy [10], the Renyi entropy [11], the linear entropy and the Wehrl entropy [12]. The Wehrl entropy has been introduced in terms of the Glauber coherent states and Husimi Q-function. In the classical limit (i.e. $\hbar \to 0$) the von Neumann entropy tends to the Wehrl entropy [13]. The Wehrl entropy has been successfully applied in the description of different properties of the quantum optical fields such as phase-space uncertainty [14, 15], quantum interference [15], decoherence [16, 17], a measure of noise [18], etc. Additionally, it has been applied to dynamical systems, e.g. the evolution of the radiation field with the Kerr-like medium [19] and with the two-level atom [17], i.e. the Jaynes–Cummings model (JCM) [20]. For the JCM it has been found that the Wehrl entropy is very sensitive to the phase-space dynamics of Q-function. Also it illustrates the loss of coherence with the upper limit for the phase randomization during the evolution of the radiation field [17]. The concept of the atomic Wehrl entropy has been developed [21] and applied to the JCM [22]. Quite recently, it has been analytically proved that the linear entropy, the von Neumann entropy and the atomic Wehrl entropy provide identical information on the entanglement in the JCM [23]. On the other hand, the concept of the phase density of the Wehrl entropy and/or the Wehrl density distribution for optical fields has been given in [18]. It has been shown that the Wehrl density distribution clearly describes: states with random phase, states with a partial phase, phase locking and phase bifurcation of quantum states of light [18]. Inspired by the concept of the Wehrl density distribution for the field we introduce—in the present paper—the marginal and density atomic Wehrl entropies for the JCM. We show that these quantities can be reduced to the information entropies, which are basically used in the treatment of the entropy squeezing [24]. Also they can provide information on the von Neumann entropy. These are interesting results motivated by the importance of the JCM in the quantum optics [20]. As is well known that the JCM can be implemented by several means, e.g. the one-atom mazer [25] and the trapped ion [26].

We perform the study in the following order. In section 2, we describe the system under consideration and derive the main relations and equations including the information entropies. In section 3, we develop the notion of the marginal atomic Wehrl entropies. In section 4, we give the explicit forms for the density atomic Wehrl entropies and discuss their connection with the information entropies.

2. Model formalism and basic relations

In this section, we give the Hamiltonian model, its wavefunction and the definition of the atomic Q-function. Additionally, we investigate the evolution of the information entropies and the von Neumann entropy.

Without the loss of generality, we restrict the attention to the simplest form of the JCM, which is the two-level atom interacting with the single cavity mode. In the rotating wave and dipole approximations the Hamiltonian governing this system is:

$$\begin{aligned} &\hat{H} = \hat{H}_0 + \hat{H}_i, \\ &\hat{H}_0 = \omega_0 \hat{a}^\dagger \hat{a} + \tfrac{1}{2}\omega_a \hat{\sigma}_z, \qquad \hat{H}_i = \lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-), \end{aligned} \tag{1}$$

where $\hat{H}_0$ $(\hat{H}_i)$ is the free (interaction) part, $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are the Pauli spin operators, ω_0 and ω_a are the frequencies of the cavity mode and the atomic transition, respectively, $\hat{a}$ $(\hat{a}^\dagger)$ is the annihilation (creation) of the cavity mode and λ is the atom–field coupling constant. In (1) we have set $\hbar = 1$ for convenience. We assume that $\omega_0 = \omega_a$ (i.e. the resonance case), the field is initially in the coherent state $|\alpha\rangle$ with real α and the atom is in the superposition of the excited and ground atomic states as:

$$|\vartheta\rangle = \cos\vartheta|e\rangle + \sin\vartheta|g\rangle, \tag{2}$$

where $|e\rangle$ $(|g\rangle)$ stands for the excited (ground) atomic state and ϑ is a phase. Under these conditions, the dynamical wavefunction of the system in the interaction picture can be expressed as:

$$|\Psi(T)\rangle = \sum_{n=0}^{\infty} [G_1(n,T)|e,n\rangle + G_2(n,T)|g,n+1\rangle], \tag{3}$$

where

$$C_n = \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{1}{2}\alpha^2\right), \qquad T = t\lambda,$$
$$G_1(n,T) = C_n \cos\vartheta \cos(T\sqrt{n+1}) - \mathrm{i}C_{n+1} \sin\vartheta \sin(T\sqrt{n+1}), \tag{4}$$
$$G_2(n,T) = C_{n+1} \sin\vartheta \cos(T\sqrt{n+1}) - \mathrm{i}C_n \cos\vartheta \sin(T\sqrt{n+1}).$$

For reasons will be made clear shortly, we give the expectation values for the atomic set operators $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ associated with the state (3) as:

$$\langle\hat{\sigma}_z(T)\rangle = \sum_{n=0}^{\infty}[|G_1(n,T)|^2 - |G_2(n,T)|^2],$$
$$\langle\hat{\sigma}_x(T)\rangle = 2\,\mathrm{Re}\sum_{n=0}^{\infty} G_1^*(n+1,T)G_2(n,T), \tag{5}$$
$$\langle\hat{\sigma}_y(T)\rangle = 2\,\mathrm{Im}\sum_{n=0}^{\infty} G_1^*(n+1,T)G_2(n,T),$$

where Re and Im stand for the real and imaginary parts of the complex quantity. Additionally, the von Neumann entropy for the JCM can be evaluated as [23]

$$\gamma(T) = -\tfrac{1}{2}[1+\eta(T)]\ln\left[\tfrac{1}{2}+\tfrac{1}{2}\eta(T)\right] - \tfrac{1}{2}[1-\eta(T)]\ln\left[\tfrac{1}{2}-\tfrac{1}{2}\eta(T)\right],$$
$$\eta(T) = \sqrt{\langle\hat{\sigma}_x(T)\rangle^2 + \langle\hat{\sigma}_y(T)\rangle^2 + \langle\hat{\sigma}_z(T)\rangle^2}. \tag{6}$$

As is well known that the von Neumann entropy is basically used for quantifying the entanglement, where $\gamma(T) = 0$ for disentangled and/or pure states and $\gamma(T) = 0.693$ for maximally entangled bipartite, i.e. $0 \leqslant \gamma(T) \leqslant \ln 2$. We conclude this part by shedding the light on the information entropies for the two-level system (i.e $N = 2$) described by the density matrix $\hat{\rho}_a$. The probability distribution of two possible outcomes of measurements of the operator $\hat{\sigma}_k$ is:

$$P_j(\hat{\sigma}_k) = \langle\psi_{kj}|\hat{\rho}_a|\psi_{kj}\rangle, \qquad j = 1, 2; \qquad k = x, y, z, \tag{7}$$

where $|\psi_{kj}\rangle$ are the eigenstates of $\hat{\sigma}_k$. In this case, the associated information entropies are:

$$H(\hat{\sigma}_k) = -\sum_{j=1}^{2} P_j(\hat{\sigma}_k)\ln P_j(\hat{\sigma}_k), \tag{8}$$

where $0 \leqslant H(\hat{\sigma}_k) \leqslant \ln 2$. It is obvious that $H(\hat{\sigma}_k)$ has the same limitations as $\gamma(T)$. It is worth mentioning that the information entropies are frequently used in the literatures, e.g., [24] in the investigation of the entropy squeezing, in particular, for systems satisfying $\langle\hat{\sigma}_z(T)\rangle = 0$. For these systems the standard uncertainty relation of the atomic operators fails to provide any useful information on the atomic system. This difficulty has been overcome using entropic the uncertainty relation [27, 28], which is related to the information entropies (8). Next, using the short-hand notations $b = \langle\hat{\sigma}_x(T)\rangle$, $c = \langle\hat{\sigma}_y(T)\rangle$, $h = \langle\hat{\sigma}_z(T)\rangle$ the relations (8) can be easily evaluated as

$$H(b) = -\frac{1}{2}(1+b)\ln\left(\frac{1}{2}+\frac{b}{2}\right) - \frac{1}{2}(1-b)\ln\left(\frac{1}{2}-\frac{b}{2}\right),$$
$$H(c) = -\frac{1}{2}(1+c)\ln\left(\frac{1}{2}+\frac{c}{2}\right) - \frac{1}{2}(1-c)\ln\left(\frac{1}{2}-\frac{c}{2}\right), \tag{9}$$
$$H(h) = -\frac{1}{2}(1+h)\ln\left(\frac{1}{2}+\frac{h}{2}\right) - \frac{1}{2}(1-h)\ln\left(\frac{1}{2}-\frac{h}{2}\right).$$

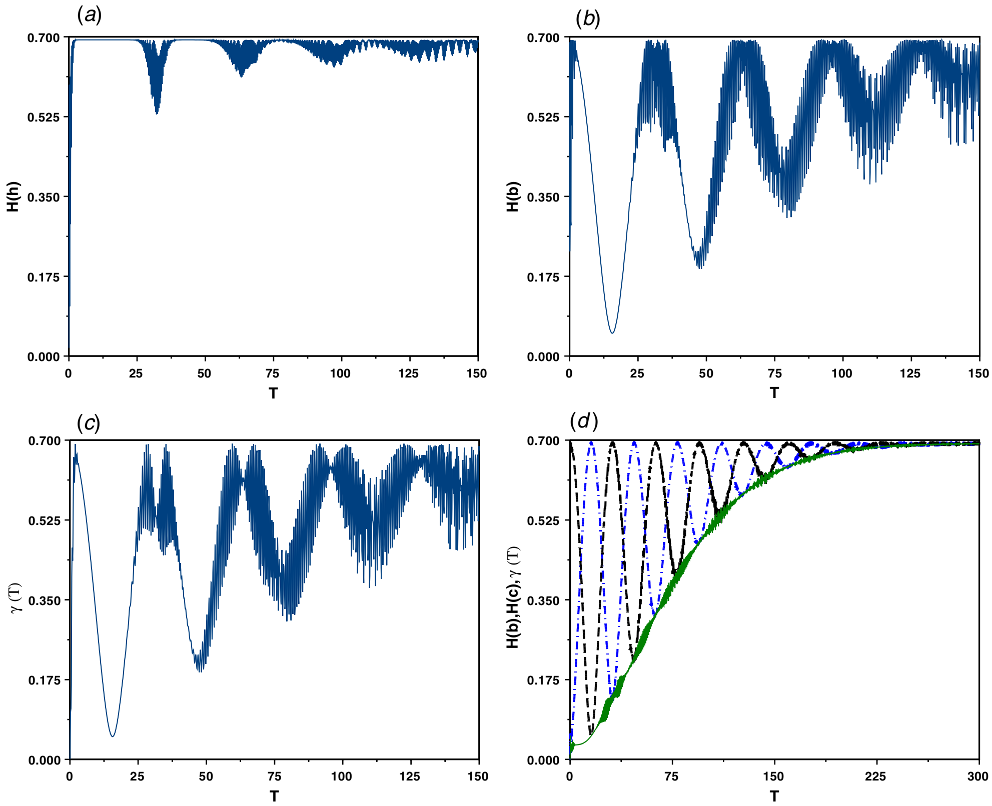

Figure 1. Evolution of the information entropies and von Nuemann entropy as indicated for $\alpha = 5$. Figures (a)–(c) and (d) are given for $\vartheta = 0$ and $\vartheta = \pi/4$, respectively. In (d) solid, dashed and dot-dashed curves are given for $\gamma(T)$, $H(c)$ and $H(b)$, respectively.

The comparison between expressions (6) and (9) shows that for particular values of the interaction parameters one of the information entropies can tend to the von Neumann entropy, e.g. when $\eta(T) \simeq |\langle\sigma_j(T)\rangle|$. To see this and to begin the discussion, we plot the von Neumann entropy and information entropies in figure 1 for given values of the interaction parameters. It is worthwhile mentioning that for $\vartheta = 0, \pi/2$ we have $b = 0$ and hence $H(b) = \ln 2$. In this case, the atomic inversion exhibits the revival-collapse phenomenon (RCP), which is remarkable in figure 1(a). One can observe that $H(h)$ provides its maximum value in the course of the collapse regions. From figures 1(b) and (c), one can realize when the atom is initially in the excited (or ground) state $\gamma(T)$ and $H(c)$ can give quite similar behavior on the bipartite. The slight difference between figures 1(b) and (c) is that the local maxima in $H(c)$ are replaced by the local minima in $\gamma(T)$. Now, the similarity between the behaviors of $\gamma(T)$ and $H(c)$ can be explained as follows. When α is real and the atom is in the excited (or ground) state we always have $\langle\hat{\sigma}_x(T)\rangle = 0$. Additionally, in the course of the collapse region we have $\langle\hat{\sigma}_z(T)\rangle = 0$; however, during the revival time the contribution of $\langle\hat{\sigma}_y(T)\rangle^2$ to $\eta(T)$ is more effective than that of $\langle\hat{\sigma}_z(T)\rangle^2$. Thus we can generally conclude that $\gamma(T) \simeq H(h)$. Now, we draw the attention to figure 1(d), which is given for $\vartheta = \pi/4$. In this case we have atomic trapping, i.e. $\langle\hat{\sigma}_z(T)\rangle \simeq 0$ and hence $H(h) \simeq \ln 2$. From figure 1(d) one can observe that $H(b)$ and $H(c)$ exhibit oscillatory behaviors and gradually show maximum values and/or long-living entanglement for large interaction times. From the solid curve in figure 1(d)

one can observe that $\gamma(T)$ is the lower envelope for $H(b)$ and $H(c)$; however, for the large interaction times $\gamma(T) = H(b) = H(c) = \ln 2$. This indicates that there is a systematic loss of coherence for longer interaction times [17]. The final remark: the above investigations will be useful in comparing these quantities with the marginal and density atomic Wehrl entropies in the following sections.

We close this section by defining the atomic Q-function $Q_a(\theta, \phi, T)$ as:

$$Q_a(\theta, \phi, T) = \frac{1}{2\pi} \langle \theta, \phi | \hat{\rho}_a(T) | \theta, \phi \rangle , \tag{10}$$

where $|\theta, \phi\rangle$ is the atomic coherent state having the form [29]:

$$|\theta, \phi\rangle = \cos(\theta/2)\,|e\rangle + \sin(\theta/2)\exp(\mathrm{i}\phi)\,|g\rangle \tag{11}$$

with $0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi$. For wavefunction (3) the atomic Q_a function can be evaluated as

$$\begin{aligned} Q_a(\theta, \phi, T) &= \frac{1}{4\pi}[1 + \beta(T)], \\ \beta(T) &= h\cos\theta + [b\cos\phi + c\sin\phi]\sin\theta. \end{aligned} \tag{12}$$

One can easily check that Q_a is normalized. The Q_a can be interpreted in the following sense. The two different spin coherent states overlap unless they are directed into two antipodal points on the Bloch sphere. This is quite different from that of the Q function of the optical field, which represents the joint probability distribution for the simultaneous (noisy) measurements of the two field quadratures [30]. From (12) it is obvious that Q_a has complete information on the set (b, c, h). In the following sections we use (12) to define the marginal and density atomic Wehrl entropies.

3. Marginal atomic Wehrl entropies

In this section, we develop the notion of the marginal atomic Wehrl entropies and show how they can tend to the information entropies (9). In doing so, we start with the definitions of the marginal atomic Q_a functions as:

$$\begin{aligned} Q_\theta &= \int_0^{2\pi} Q_a(\theta, \phi, T)\,\mathrm{d}\phi, \\ Q_\phi &= \int_0^{\pi} Q_a(\theta, \phi, T)\sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{13}$$

From (12) and (13) one can easily obtain

$$\begin{aligned} Q_\theta &= \frac{1}{2}(1 + h\cos\theta), \\ Q_\phi &= \frac{1}{2\pi}\left[1 + \frac{\pi}{4}(b\cos\phi + c\sin\phi)\right]. \end{aligned} \tag{14}$$

It is obvious that Q_θ (Q_ϕ) includes information on $\langle\hat{\sigma}_z(T)\rangle$ $(\langle\hat{\sigma}_x(T)\rangle, \langle\hat{\sigma}_y(T)\rangle)$. Now we are in a position to define the marginal atomic Wehrl entropies as:

$$\begin{aligned} W_\theta(T) &= -\int_0^{\pi} Q_\theta \ln Q_\theta \sin\theta\,\mathrm{d}\theta, \\ W_\phi(T) &= -\int_0^{2\pi} Q_\phi \ln Q_\phi\,\mathrm{d}\phi. \end{aligned} \tag{15}$$

As W_θ and W_ϕ have been evaluated from the θ and ϕ components of Q_a we call them marginal atomic Wehrl entropies. Nevertheless, they are phase independent. It is obvious that the quantities $W_\phi(T)$ and $W_\theta(T)$ have the notion of the entropy, where Q_ϕ and Q_θ are always non-negative quantities (cf (14)). In this context, $W_\phi(T)$ and $W_\theta(T)$ can be interpreted as being information measures associated with the components $\langle\hat{\sigma}_z(T)\rangle$ and $(\langle\hat{\sigma}_x(T)\rangle, \langle\hat{\sigma}_y(T)\rangle)$, respectively. Substituting (14) into (15) and carrying out the integration we obtain:

$$\begin{aligned} W_\theta(T) &= \ln(2\sqrt{e}) + \frac{(1-h)^2}{4h}\ln(1-h) - \frac{(1+h)^2}{4h}\ln(1+h), \\ &= H(h) + \frac{1}{2} + \frac{(1-h^2)}{4h}\ln\left[\frac{1-h}{1+h}\right], \end{aligned} \tag{16}$$

$$\begin{aligned} W_\phi(T) &= \ln(2\pi) - \sum_{n=0}^{\infty}\frac{(2n)!}{4^{n+1}[(n+1)!]^2}\xi^{n+1}, \\ &= \ln(2\pi) - \xi\,{}_3F_2\left(\left\{\frac{1}{2},1,1\right\},\{2,2\};\xi\right) \\ &= \ln(2\pi) - 1 + \sqrt{1-\xi} - \ln\left[\frac{1+\sqrt{1-\xi}}{2}\right], \end{aligned} \tag{17}$$

where $\xi = \frac{\pi^2(b^2+c^2)}{16}$ and ${}_qF_p(\{\tau_1,\tau_2,\ldots,\tau_q\},\{\upsilon_1,\upsilon_2,\ldots,\upsilon_p\};\xi)$ is the generalized hypergeometric function [37]. In the derivation of (17) we have used the series expansion of the logarithmic function and the following integral identity [37]:

$$\int_0^{2\pi}(c_1\sin x + c_2\cos x)^k\,\mathrm{d}x = \begin{cases} 0 & \text{for} \quad k = 2m+1, \\ 2\pi\frac{(2m)!}{4^m(m!)^2}\left(c_1^2+c_2^2\right)^m & \text{for} \quad k = 2m, \end{cases} \tag{18}$$

where c_1, c_2 are the c-numbers and k is a positive integer. The second and the third lines of (17) include different forms for the summation in the first line.

From the extreme values of h, b, c and from expressions (16) and (17) one can obtain the following inequalities:

$$\tfrac{1}{2} \leqslant W_\theta(T) \leqslant \ln 2, \qquad \ln(2\pi) - 0.17 \leqslant W_\phi(T) \leqslant \ln(2\pi). \tag{19}$$

The number 0.17 is the value of the series in the first line of (17), which has been obtained from its exact form in the third line. We plot (16) and (17) in figures 2 for the given values of the interaction parameters. Comparing parts (a) and (b) in figure 1 with those in figure 2 leads to—apart from the different scales in figures 1 and 2—when the atom is in the excited (or ground) W_θ and W_ϕ can give information on $H(h)$ and $H(c)$, respectively. Nevertheless, when $\langle\hat{\sigma}_z(T)\rangle \simeq 0$ (i.e. $\vartheta = \pi/4$) we have $H(h) = W_\theta = \ln 2$; however, W_ϕ gives information on $\gamma(T)$ (compare the solid curve in figure 1(d) with figure 2(c)). It is obvious that W_ϕ stabilizes at a certain level after a sufficient long interaction time. In the language of entanglement, when $W_\phi(T) = \ln(2\pi) - 0.17$ [or $\ln(2\pi)$] the bipartite is disentangled [or maximally entangled]. Next, we treat the problem of different scales between the marginal atomic Wehrl entropies and the information entropies. This can be raised by redefining W_θ and W_ϕ to have the limitations of the corresponding information entropies, i.e. $0 \leqslant H(.) \leqslant \ln 2$. With this in mind and from (19) we obtain

$$\begin{aligned} \widehat{W}_\theta(T) &= \frac{\ln 2}{\ln\left(\frac{4}{e}\right)}[2W_\theta(T) - 1], \\ W(T) &= \frac{\ln 2}{\ln(2\pi) - 0.17}[W_\phi(T) - 0.17]. \end{aligned} \tag{20}$$

We close this section by checking the validity of (20). As an example, we have plotted the rescaled quantity W in figure 2(d). The comparison between figures 1(d) and 2(d) is instructive and shows that $W(T) \simeq \gamma(T)$.

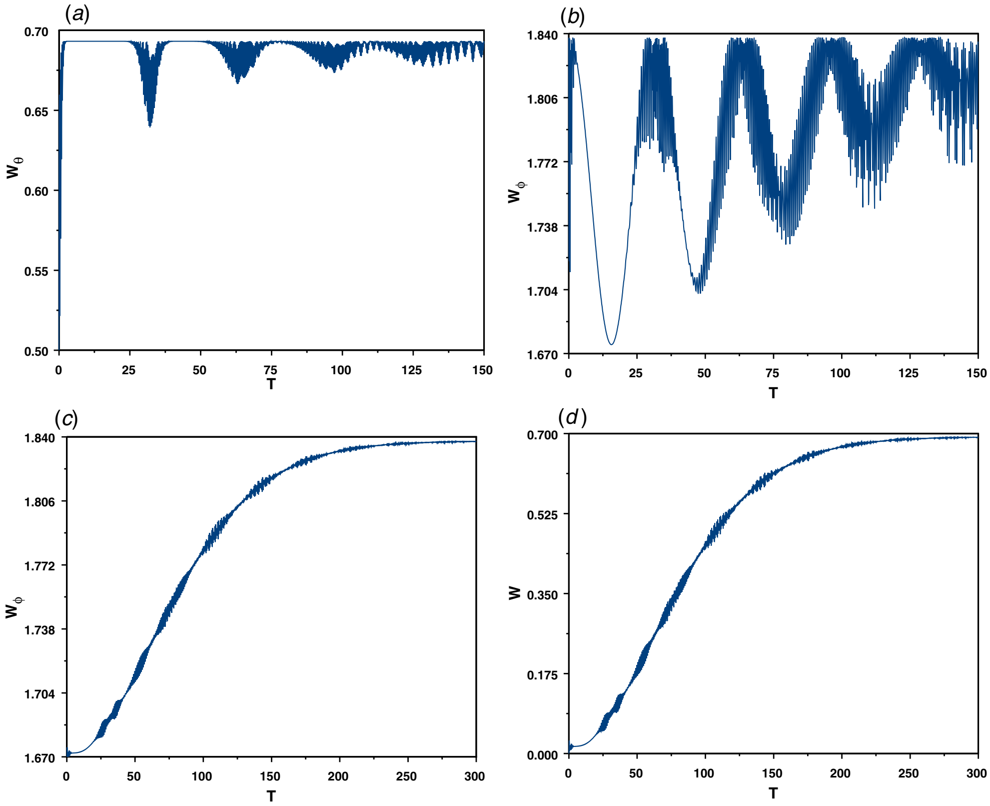

Figure 2. Evolution of the marginal atomic Wehrl entropies as indicated in the figures for $\alpha = 5$ against the scaled time T. Figures (*a*), (*b*) and (*c*), (*d*) are given for $\vartheta = 0$ and $\pi/4$, respectively.

4. Density atomic Wehrl entropies

In this section we derive the explicit expressions for the density atomic Wehrl entropies, which have been numerically treated, e.g. [22] in the static regime. Moreover, we deduce the connections between these quantities and the information entropies. The density atomic Wehrl entropies can be defined as

$$\begin{aligned} Z_\theta(T) &= -\int_0^{2\pi} Q_a(\theta,\phi,T)\ln Q_a(\theta,\phi,T)\,\mathrm{d}\phi, \\ Z_\phi(T) &= -\int_0^{\pi} Q_a(\theta,\phi,T)\ln Q_a(\theta,\phi,T)\sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{21}$$

It is evident that Z_θ, Z_ϕ are phase dependent and they have the notion of the entropy. The components Z_θ and Z_ϕ can be interpreted as being the information measures associated with the directions θ and ϕ, respectively. In this respect, they may also be called geometric information entropies. Substituting (12) into (21) and carrying out the integration we obtain the following expressions:

$$
\begin{aligned}
Z_\theta(T) &= (1 + h\cos\theta)\frac{\ln(4\pi)}{2}\\
&\quad - \frac{1}{2}\left\{h\cos\theta + \sum_{n=2}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\frac{(-1)^n(n-2)!}{(n-2r)!(r!)^2 4^r}(h\cos\theta)^{n-2r}\sin^{2r}\theta(b^2+c^2)^r\right\},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
Z_\phi(T) &= \frac{1}{4\pi}\left[2 + \frac{\pi\varepsilon}{2}\right]\ln(4\pi) - \frac{\varepsilon}{8}\\
&\quad + \sum_{n=1}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(2n-1)!(n-r)!(-1)^s h^{2(n-r)}\varepsilon^{2r+1}}{(2r+1)!(2n-2r)!(n-r-s)!s!(2s+2r+3)4^{s+r+2}\beta(s+r+2,s+r+2)}\\
&\quad - \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{r}\frac{(2n-2)!r!(-1)^s h^{2(n-r)}\varepsilon^{2r}}{(2r)!(2n-2r)!(r-s)!s!(2n+2s-2r+1)},
\end{aligned}
\tag{23}
$$

where $\beta(.)$ is the Beta function and $\varepsilon = b\cos\phi + c\sin\phi$. In the derivation of (22) and (23), we have used procedures similar to those done for (17) as well as for the following identity [37]:

$$
\int_0^\pi \sin^{m-1}x\,\mathrm{d}x = \frac{\pi}{2^{m-1}m\beta\left(\frac{m+1}{2},\frac{m+1}{2}\right)}.
\tag{24}
$$

From (22) and (23) one can realize that each of Z_θ and Z_ϕ can give information on the atomic components, i.e. h, b, c. This is in contrast to the marginal atomic Wehrl entropies (cf (16) and (17)). Also their limitations are sensitive to the phase as well as the initial atomic states. We have numerically checked this fact.

Next, we show how Z_θ and Z_ϕ can be connected with the information entropies as well as $\gamma(T)$. For instance, throughout straightforward calculations one can easily show:

$$
Z_{\theta=0}(T) + Z_{\theta=\pi}(T) = H(h) + \ln(2\pi),
\tag{25}
$$

$$
\begin{aligned}
Z_{\theta=\pi/2}(T) &= \frac{1}{2}\ln(4\pi) - \frac{1}{8}\sum_{n=0}^{\infty}\frac{(2n)!\xi^{n+1}}{4^n[(n+1)!]^2}\\
&= \frac{1}{2}\ln(4\pi) - \frac{1}{2} + \frac{1}{2}\sqrt{1-\xi} - \frac{1}{2}\ln\left[\frac{1+\sqrt{1-\xi}}{2}\right],
\end{aligned}
\tag{26}
$$

where $\xi = b^2 + c^2$. The series in the first line of (26) is similar to that in (17). Thus the comparison between (17) and (26) shows that $Z_{\theta=\pi/2}(T)$ can carry information on the von Neumann entropy. To be more specific, from (25) we can obtain the following rescaled density atomic Wehrl entropy:

$$
\widehat{Z}_{\theta=\pi/2}(T) = \frac{\ln 2}{0.15}\left[Z_{\theta=\pi/2}(T) - \frac{1}{2}\ln(4\pi) + 0.15\right],
\tag{27}
$$

where the number 0.15 is obtained from (26) using the extreme values of b, c. We have numerically found that $\widehat{Z}_{\theta=\pi/2}(T) \simeq \gamma(T)$. Now, we draw the attention to Z_ϕ. When $\varepsilon \to 0$ (i.e. for $b = 0$ and $\phi = 0$) expression (23) reduces to

$$
Z_\phi(T) = \frac{1}{2\pi}\left\{\ln(2\pi) + H(h) + \frac{1}{2} + \frac{(1-h^2)}{4h}\ln\left[\frac{1-h}{1+h}\right]\right\}.
\tag{28}
$$

Also when $h \simeq 0$ (i.e. the atomic trapping case) expression (23) can give information on b or c based on the value of ϕ.

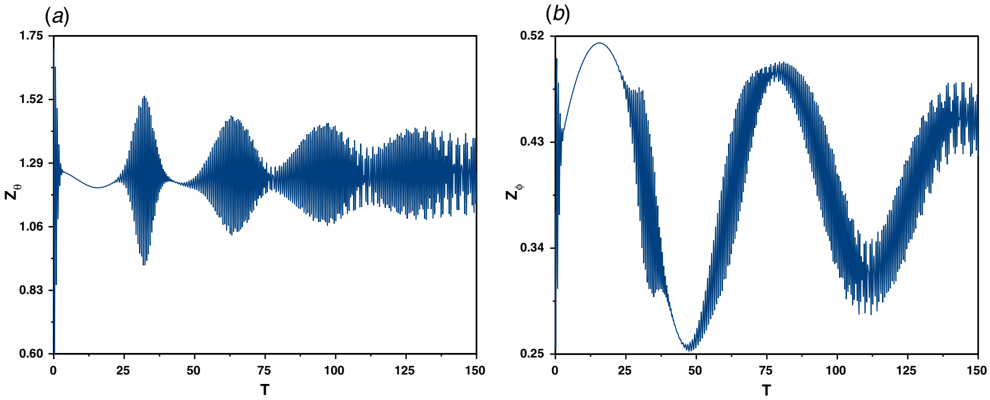

Figure 3. Evolution of the density atomic Wehrl entropies as indicated in the figures for $(\alpha, \vartheta) = (5, 0)$ and $\theta = \phi = \pi/4$ against the scaled time T.

We close this section by studying numerically the case for which two or all of the components (b, c, h) give comparable contribution to the density atomic Wehrl entropies (see figures 3). In these figures we have taken $\theta = \phi = \pi/4, \vartheta = 0$. It is obvious that in the evolution of $Z_{\theta=\pi/4}$ $(Z_{\phi=\pi/4})$ the behavior of $\langle\sigma_z(T)\rangle$ $(\langle\sigma_y(T)\rangle)$ is dominant. It seems that this is related to the leading terms in the expressions (22) and (23).

In conclusion, in this paper we have developed the notion of the marginal and density atomic Wehrl entropies for the JCM. We have shown that there are relationships between these quantities and both of the information entropies and von Neumman entropy. The marginal (density) atomic Wehrl entropies are phase independent (dependent) and have (do not have) clear limitations. Furthermore, the marginal (density) atomic Wehrl entropies can be used as the information measures associated with the atomic components (orientations θ and ϕ). Finally, we have derived various analytical relations.

Acknowledgments

I would like to thank the first referee, who I do not know, for his fruitful comments. He found an error and corrected. The closed forms in (17) and (26) were given by him.

References

[1] Benenti G, Casati G and Strini G 2005 *Principle of Quantum Computation and Information* (Singapore: World Scientific)
[2] Bennet C H, Brassard G, Crepeau C, Jozsa R, Peresand A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[3] Glöckl O, Lorenz S, Marquardt C, Heersink J, Brownnutt M, Silberhorn C, Pan Q, Loock P V, Korolkova N and Leuchs G 2003 *Phys. Rev.* A **68** 012319
Yang M, Song W and Cao Z-L 2005 *Phys. Rev. A* **71** 034312
Li H-R, Li F-L, Yang Y and Zhang Q 2005 *Phys. Rev.* A **71** 022314
[4] Peng C-Z *et al* 2005 *Phys. Rev. Lett.* **94** 150501
[5] Volz U, Weber M, Schlenk D, Rosenfeld W J, Vrana J, Saucke K, Kurtsiefer C and Weinfurter H 2006 *Phys. Rev. Lett.* **96** 030404
[6] Zhao Z, Chen Y-A, Zhang A-N, Yang T, Briegel H and Pan J-W 2004 *Nature* **430** 54
[7] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2007 *Preprint* quant-ph/0702225

[8] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[9] Vedral V 2002 *Rev. Mod. Phys.* **74** 197
[10] Bastiaanns M J 1984 *J. Opt. Soc. Am.* A **1** 711
Tsallis C 1988 *J. Stat. Phys.* **55** 479
[11] Renyi A 1970 *Probability Theory* (Amsterdam: North Holland)
[12] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
Wehrl A 1991 *Rep. Math. Phys.* **30** 119
[13] Beretta G P 1984 *J. Math. Phys.* **25** 1507
[14] Bužek V, Keitel C H and Knight P L 1995 *Phys. Rev.* A **51** 2575
Vaccaro J A and Orlowski A 1995 *Phys. Rev.* A **51** 4172
Watson J B, Keitel C H, Knight P L and Burnett K 1996 *Phys. Rev.* A **54** 729
[15] Bužek V, Keitel C H and Knight P L 1995 *Phys. Rev.* A **51** 2594
[16] Anderson A and Halliwell J J 1993 *Phys. Rev.* D **48** 2753
[17] Orlowski A, Paul H and Kastelewicz G 1995 *Phys. Rev.* A **52** 1621
[18] Miranowicz A, Matsueda H and Wahiddin M R B 2000 *J. Phys. A: Math. Gen.* **33** 51519
[19] Jex I and Orlowski A 1994 *J. Mod. Opt.* **41** 2301
[20] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[21] Zyczkowski K 2001 *Physica* E **9** 583
[22] Obada A-S and Abdel-Khalek S 2004 *J. Phys. A: Math. Gen.* **37** 6573
El-Orany F A A, Abdel-Khalek S, Abd-Aty M and Wahiddin M R B 2007 *International J. Theor. Phys.* (at press) (*Preprint* quant-ph/0703043)
[23] El-Orany F A A 2007 *Preprint* quant-ph/07042347
[24] Fang M-F, Zhou P and Swain S 2000 *J. Mod. Opt.* **47** 1043
[25] Rempe G, Walther H and Klein N 1987 *Phys. Rev. Lett.* **57** 353
[26] Vogel W and De Matos Filho R L 1995 *Phys. Rev.* A **52** 4214
[27] Hirschman I I 1957 *Am. J. Math.* **79** 152
Bialynickibirula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
Beckner W 1975 *Ann. Math.* **102** 159
Deutsch D 1983 *Phys. Rev. Lett.* **50** 631
[28] Maasen H and Unk J B M 1988 *Phy. Rev. Lett.* **60** 1103
Garret A J M and Gull S F 1990 *Phys. Lett.* A **151** 453
Sanchez-Ruiz J 1993 *Phys. Lett.* A **173** 233
[29] Vieira V R and Sacramento P D 1995 *Ann. Phys. (NY)* **242** 188
[30] Leonhardt U and Paul H 1993 *J. Mod. Opt.* **40** 1745
[31] Ekert A 1991 *Phys. Rev. Lett.* **67** 661
Cirac J I and Gisin N 1997 *Phys. Lett.* A **229** 1
Fuchs C A, Gisin N, Griffiths R B, Niu C-S and Peres A 1997 *Phys. Rev.* A **56** 1163
[32] Ye L and Guo G-C 2005 *Phy. Rev.* A **71** 034304
Mozes S, Oppenheim J and Reznik B 2005 *Phys. Rev.* A **71** 012311
[33] Lambert N, Emary C and Brandes T 2004 *Phys. Rev. Lett.* **92** 073602
[34] Vedral V 2004 *New. J. Phys.* **6** 102
[35] Dür W, Hartmann L, Hein M, Lewenstein M and Briegel H J 2005 *Phys. Rev. Lett.* **94** 097203
[36] Shannon C E 1948 *Bell Syst. Tech. J.* **27** 379
[37] Gradshteyn S and Ryzhik I M 1994 *Table of Integrals, Series, and Products* ed Jeffrey A 5th edn (New York: Academic) p 424, 416